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LETTER TO THE EDITOR

Integrals of motion for the Lorenz system

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Abstract. Three new cases when the Lorenz system has (time dependent) integrals of motion are given.

The integrability of the Lorenz system (Lorenz 1963)

$$\mathrm{d}x/\mathrm{d}t=\sigma y-\sigma x, \qquad \mathrm{d}y/\mathrm{d}t=-y-xz+rx, \qquad \mathrm{d}z/\mathrm{d}t=xz-bz \tag{1}$$

was recently investigated in connection with the analytic properties of its solutions (Segur 1980, Tabor and Weiss 1981). Demanding the Painlevé property of the solutions (Ablowitz *et al* 1980), it is possible to identify the cases when (1) has one or more integrals of motion and is (at least partially) integrable (Segur 1980, Tabor and Weiss 1981). In addition to the linear case $\sigma=0$ system, (1) is integrable for $\sigma=\frac{1}{2}$, $b=1$, $r=0$ (in this case it has two integrals of motion) and has one integral of motion for $\sigma=1$, $b=2$, $r=\frac{1}{9}$ and $\sigma=\frac{1}{3}$, $b=0$, r arbitrary. On the other hand two other cases are known, not related to the Painlevé property, when the integrals of motion exist (Segur 1980, Tabor and Weiss 1981, Steeb 1982): $b=1$, $r=0$, σ arbitrary and $b=2\sigma$, r arbitrary. Tabor and Weiss tried to relate these cases to other analytic properties of solutions and made a conjecture about other possible cases for which the integrals of motion exist.

In this Letter I give all the values of the parameters σ, b and r for which system (1) has the integral of motion of the form (cf Steeb 1982)

$$F(x,y,z,t)=W(x,y,z)\exp(-\lambda t) \tag{2}$$

where $W(x, y, z)$ is the polynomial of order less than five:

$$W(x,y,z)=\sum_{k+l+m\leqslant s} A_{klm}x^k y^l z^m, \qquad s=4. \tag{3}$$

It can be observed that the constant λ must be equal to

$$\lambda=k\lambda_1+l\lambda_2+m\lambda_3, \qquad k+l+m\leqslant s, \qquad k,l,m=0,1,2,\ldots \tag{4}$$

where λ_1, λ_2, λ_3 are the eigenvalues of the version of (1) which is linearised around $(0,0,0)$.

Indeed, using the method of Carleman embedding (Steeb and Wilhelm 1980, Andrade and Rauh 1981), the Lorenz system can be rewritten in the form

$$\mathrm{d}p/\mathrm{d}t=Mp \tag{5}$$

where p is the infinite-dimensional vector built up of the quantities $p_{lkm}=x^k y^l z^m$ arranged in the following manner (Andrade and Rauh 1981). For a given n we build the vector $(p_{n1}, p_{n2}, \ldots, p_{nK(n)})$, $K(n)=\frac{1}{2}(n+1)(n+2)$ consisting of all p_{lmk} with $l+k+m=n$. This gives the correspondence $p_{lmk}\to p_{ns}$, $1\leqslant s\leqslant K(n)$. Now $p=(p_{11}, p_{12}, p_{13}, \ldots, p_{n1}, p_{n2}, \ldots p_{nK(n)}, \ldots)^{\mathrm{T}}$. According to Steeb and Wilhelm (1980) and Andrade and Rauh (1981) matrix M has a block structure:

$$M=\begin{bmatrix} D_1 & N_1 & 0 & 0\ldots \\ 0 & D_2 & N_2 & 0\ldots \\ 0 & 0 & D_3 & N_3\ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where D_s $(s=1,2,3,\ldots)$ are $K(s)\times K(s)$ matrices with the eigenvalues given by the formula (4).

From (2) and (3) we have

$$F(x, y, z, t)=\sum_{k+l+m\leqslant s} A_{klm}p_{klm}\exp(-\lambda t)=\sum_{\substack{k=1,\ldots,s\\ r=1,\ldots,K(s)}} A_{kr}p_{kr}\exp(-\lambda t). \tag{6}$$

Differentiating, using (6) truncated to the dimension $K(1)+K(2)+\ldots+K(s)$ and equating to zero the coefficients of different p_{kr} we arrive at

$$A(M_{(s)}-\lambda)=0 \tag{7}$$

where $M_{(s)}$ is the truncated matrix and $A=(A_{11}, A_{12}, A_{13}, \ldots, A_{s1}, \ldots, A_{sK(s)})$. Equation (7) has non-zero solutions only when λ is an eigenvalue of $M_{(s)}$ i.e. $\lambda=k\lambda_1+l\lambda_2+m\lambda_3$, $k+l+m\leqslant s$.

Certainly (4) is only a necessary condition because, in addition to (7), we have the extra equations for A_{klm} obtained by equating to zero the coefficients in $(\mathrm{d}/\mathrm{d}t)F(x, y, z, t)$ of the terms containing p_{klm} with $k+l+m=s+1$.

For $s=4$ we obtain the following cases for which the equations for A_{klm} have non-zero solutions and the integrals of motion exist:

(1) $b=2\sigma$, r arbitrary, $F=(x^2-2\sigma z)\exp(2\sigma t)$
(2) $b=0$, $\sigma=\frac{1}{3}$, r arbitrary, $F=(-rx^2+\frac{1}{3}y^2+\frac{2}{3}xy+x^2z-\frac{3}{4}x^4)\exp(\frac{4}{3}t)$
(3) $b=1$, $r=0$, σ arbitrary, $F=(y^2+z^2)\exp(2t)$
(4) $b=4$, $\sigma=1$, r arbitrary, $F=(4(1-r)z+rx^2+y^2-2xy+x^2z-\frac{1}{4}x^4)\exp(4t)$
(5) $b=1$, $\sigma=1$, r arbitrary, $F=(-rx^2+y^2+z^2)\exp(2t)$
(6) $b=6\sigma-2$, $r=2\sigma-1$, $F=\{[(2\sigma-1)^2/\sigma]x^2+y^2-(4\sigma-2)xy+x^2z-(1/4\sigma)x^4\}\exp(4\sigma t)$.

Cases (1), (2) and (3) are known in the literature (Segur 1980, Tabor and Weiss 1981, Steeb 1982), whereas the remaining three seem to be new ones. It is worth stressing that none of the integrals (4), (5) and (6) is connected with the conjectures given by Tabor and Weiss based on the analytic properties of solutions.

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